# Perturbation theory for Lyapunov exponents of an Anderson model on a strip

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**Abstract**

It is proven that the inverse localization length of an Anderson model on a strip of width $L$ is bounded above by $L/\lambda^2$ for small values of the coupling constant $\lambda$ of the disordered potential. For this purpose, a formalism is developed in order to calculate the bottom Lyapunov exponent associated with random products of large symplectic matrices perturbatively in the coupling constant of the randomness.

## 1 Main result and discussion

The Anderson model describes electronic waves scattered at random obstacles. Here the physical space is supposed to be quasi one dimensional and given by an infinite strip of finite but large width $L$. Hence the Hilbert space is $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ and states therein will be decomposed as $\psi = (\psi(n))_{n\in\mathbb{Z}}$ with $\psi(n) \in \mathbb{C}^L$. The Anderson Hamiltonian on a strip is then defined by

$$(H_L(\lambda)\psi)(n) \;=\; -\psi(n+1) - \psi(n-1) + \Delta_L\psi(n) + \lambda V(n)\psi(n)\ .$$

Here $\Delta_L : \mathbb{C}^L \to \mathbb{C}^L$ is the transverse (one dimensional) discrete laplacian with periodic boundary conditions; denoting the cyclic shift on $\mathbb{C}^L$ by $S$, it is given by $\Delta_L = -S - S^*$. For $L = 1, 2$, one may rather set $\Delta_1 = 0$ and $\Delta_2 = -S = -S^*$, but our main interest will be in the case $L \geq 3$ anyway. Furthermore, $\lambda \in \mathbb{R}$ is the coupling constant of the random potential $V(n) : \mathbb{C}^L \to \mathbb{C}^L$ which is a diagonal matrix $V(n) = \mathrm{diag}(v(n,1), \ldots, v(n,L))$. All real numbers $(v(n,l))_{n\in\mathbb{Z},\, l=1,\ldots,L}$ are independent and identically distributed centered real variables with unit variance. Given a fixed energy $E \in \mathbb{R}$, it is convenient to rewrite the eigenvalue equation $H_L(\lambda)\psi = E\psi$ using the transfer matrices

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} \;=\; T(n) \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}\ , \qquad T(n) \;=\; \begin{pmatrix} \Delta_L + \lambda V(n) - E\,\mathbf{1} & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\ . \tag{1}$$

This work concerns the study of the asymptotics of their random products which is characterized by the Lyapunov exponents. One way [BL, CL] to define these exponents is to use a formalism of second quantization. For $p = 1, \ldots, L$, let $\Lambda^p\mathbb{C}^{2L}$ denote the Hilbert space of the anti-symmetrized $p$-fold tensor products of $\mathbb{C}^{2L}$, the scalar product being given via the determinant. Given a linear map $T$ on $\mathbb{C}^{2L}$, its second quantized $\Lambda^pT$ on $\Lambda^p\mathbb{C}^{2L}$ is defined as usual. Now the whole family of non-negative Lyapunov exponents $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_L \geq 0$ are defined by:

$$\sum_{l=1}^{p} \gamma_l \;=\; \lim_{N\to\infty} \frac{1}{N}\, \mathbf{E} \log\left( \left\| \prod_{n=1}^{N} \Lambda^p T(n) \right\| \right) , \qquad p \;=\; 1, \ldots, L , \tag{2}$$

where the expectation is taken over all random variables. Positivity of the bottom Lyapunov exponent $\gamma_L$ was already known [BL, GM, GR] and is sufficient to assure Anderson localization [KLS, BL]. The object of this work is to provide a quantitative lower bound for small coupling of the random potential.

We will need a hypothesis excluding exceptional energies with Kappus-Wegner breakdown of the perturbation theory to leading order [KW]. This can in principle be overcome [KW, BK, CK]. Let $[c]$ denote the integer part of $c \in \mathbb{R}$.

**Main hypothesis:** Let $\mu_l = -2\cos(\frac{2\pi l}{L}) - E$ and, if $|\mu_l| \leq 2$, set $e^{\imath\eta_l} = \frac{1}{2}(\mu_l + \imath\sqrt{4-\mu_l^2})$. Except if $\sigma = -1$ and $j = k$ or $\{k, l\} = \{m, j\}$, we suppose

$$e^{\imath(\eta_k+\sigma\eta_j)} \neq 1 , \qquad e^{\imath(\eta_k+\eta_l+\sigma\eta_m+\sigma\eta_j)} \neq 1 , \qquad k, l, m, j = 1, \ldots, \left[\frac{L+1}{2}\right] , \quad \sigma = \pm 1 . \tag{3}$$

**Theorem 1** *Suppose $E \in \mathbb{R}$ is in the spectrum of $H_L(0)$ and satisfies the Main hypothesis.*
(i) *Then*

$$\gamma_L \;\geq\; \frac{\lambda^2}{8L} \;+\; \mathcal{O}(\lambda^3) .$$

(ii) *Let $E_b$ be a band edge of $H_L(0)$ and $E = E_b + \epsilon$ be in the spectrum of $H_L(0)$. Then*

$$\gamma_L \;\geq\; \frac{\lambda^2}{8L}\,\frac{1}{|\epsilon|} \;+\; \mathcal{O}(\lambda^3) .$$

For the case $L = 1$, this was proven by Pastur and Figotin [PF] and in a related situation of hamiltonian stochastic differential equations by Arnold, Papanicolaou and Wihstutz [APW]. Actually, we go beyond the above theorem and prove an asymptotic formula for $\gamma_L$ in Theorem 3 in Section 4.6 below. We then argue (non-rigorously) in Section 5 that the bound (i) gives the right order of magnitude for all energies away from the band edges and the so-called internal band edges which are defined by the property that $\eta_l = 0$ for some $l$. Near the band edges, the bottom Lyapunov exponent is much larger according to item (ii). Indications for such stronger localization properties in this regime appeared also in [Klo]. It is straightforward to analyse the large deviations of the growth behavior of the transfer matrices around the typical

behavior given by the Lyapunov exponent with the techniques of [JSS, Section 5]. If one adds the supplementary hypothesis that $\mathbf{E}(v(n,l)^3) = 0$, then the corrections are actually of the order $\mathcal{O}(\lambda^4)$. The main deficiency of the present work is the lack of control of the error term on the strip with $L$ and the energy $E$.

The method of proof transposes directly to a more abstract setting of random products of symplectic matrices with small coupling, if only one supposes that the modulus one eigenvalues of the unperturbed part are non-degenerate (in the language of Section 2.3, elliptic channels are then non-degenerate). In order to deal with the degeneracies appearing in the Anderson model, the concrete form of the random perturbation in (1) is however heavily used. We believe that hamiltonian stochastic differential equations could also be treated. Preliminary results in this framework were obtained by Teichert [Tei].

Let us briefly describe the key steps of the proof. First the transfer matrix at $\lambda = 0$ is diagonalized into symplectic blocks given by rotations (Sections 2.1 and 2.5). Then the matrix elements of the random perturbation are calculated in that representation (Section 2.2). This normal form allows to derive a basic perturbative formula for the Lyapunov exponents (Sections 4.2 and 4.6). A new ingredient herein is the consistent use of symplectic frames. It is then possible to apply a crucial identity related to the geometry of Lagrangian manifolds (Lemma 2 in Section 3.1). The normal form of the transfer matrix now allows to efficiently control the oscillatory sums appearing in the perturbative formula (Section 4.3). There is an inessential technical difficulty due to the presence of so-called hyperbolic channels. They do not appear if $L$ is odd and $E$ is near the band center. The text is written such that the reader can understand this case and hence the main point of the argument by skipping Section 3.5 and then omitting Sections 4.5 and 4.6.

In order to compare Theorem 1 with results in the physics literature, let us interprete $L$ as the number of channels of a disordered wire and $\gamma_L$ as the associated inverse localization length. Then the behaviour

$$\gamma_L \sim \frac{\lambda^2}{L} + \mathcal{O}(\lambda^3) \tag{4}$$

confirms the predictions of Thouless [Tho] as well as the Dorokhov-Mello-Pereyra-Kumar theory (see [Ben] for a review on the latter).

For the Anderson model in two dimensions, all waves are expected to localize even at small disorder [AALR]. Few rigorous results indicating such a phenomenon are known. Even though one may think of the strip as an approximation to the two dimensional situation, it is unlikely that (4) gives much insight. For a proof of localization, one would need to prove the so-called initial length scale estimate in order to apply the multiscale analysis [FS, DK, GK]. It states that the wave functions on a square of appropriate diameter decrease from center to boundary (or inversely) with a high probability. But even if the error term in (4) could be neglected at small $\lambda$, exponential decay of typical eigenfunctions is noticeable only on a length scale $N$ given by $N\gamma_L = \mathcal{O}(1)$, that is $N = L/\lambda^2$, which is much larger than the strip width $L$. Therefore (4) is of interest only in the quasi one dimensional situation. Indeed, Anderson localization in two dimensions is expected to be a non-perturbative phenomenon (like BCS theory) and

thus not tractable by a "naive perturbation theory" as developed here. This is reflected by the prediction that the 2D localization length behaves non-analytically like $e^{1/\lambda^2}$ for small $\lambda$ [AALR]. Rigorously known is only a lower bound on the phase-space localization of the eigenfunctions [SSW].

# 2 Analysis of the transfer matrix

Each transfer matrix $T(n)$ is a random element of the symplectic group

$$\mathrm{SP}(2L,\mathbb{R}) \;=\; \left\{ T \in \mathrm{M}_{2L\times 2L}(\mathbb{R}) \;\middle|\; T^t J T = J \right\} \;, \qquad J \;=\; \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \;.$$

The aim of this section is to construct a symplectic basis transformation $M \in \mathrm{SP}(2L,\mathbb{R})$ such that

$$M^{-1}\, T(n)\, M \;=\; R\,(\mathbf{1} - \lambda\, P(n)) \;, \tag{5}$$

where the free transfer matrix $R$ (*i.e.* $\lambda = 0$) takes a particularly simple form given by a direct sum of rotations. The random perturbation $P(n)$ lies in the Lie algebra $\mathrm{sp}(2L,\mathbb{R})$. Its matrix elements and some of their expectation values will be calculated below. Throughout this section the index $n$ is kept fixed and will thus be suppressed.

## 2.1 Normal form without disorder

Let us introduce, for $l = 1,\ldots,L$,

$$\phi_l = \begin{pmatrix} \phi_l(1) \\ \vdots \\ \phi_l(L) \end{pmatrix} \in \mathbb{C}^L \;, \qquad \phi_l(k) \;=\; \frac{1}{\sqrt{L}}\, \exp\left(\frac{2\pi\imath\, lk}{L}\right) \;.$$

Then $\Delta_L \phi_l = -2\cos(2\pi\, l/L)\,\phi_l$. Note that the fundamental $\Phi_L = \phi_L$ is real; moreover, for even $L$, the vector $\Phi_{L/2} = \phi_{L/2}$ is real as well. For other $l$, real normalized eigenvectors are obtained by

$$(\Phi_l, \Phi_{L-l}) \;=\; (\phi_l, \phi_{L-l})\, \frac{1}{\sqrt{2}} \begin{pmatrix} -\imath & 1 \\ \imath & 1 \end{pmatrix} \;.$$

Next define an orthogonal matrix $m \in \mathrm{O}(L,\mathbb{R})$ and unitaries $d, f \in \mathrm{U}(L,\mathbb{C})$ by

$$m \;=\; (\Phi_1,\ldots,\Phi_L) \;, \qquad f \;=\; (\phi_1,\ldots,\phi_L) \;, \qquad m \;=\; f\, d \;.$$

Finally introduce the diagonal matrix $\mu = \mathrm{diag}(\mu_1,\ldots,\mu_L)$ where $\mu_l = -2\cos(2\pi\, l/L) - E$. With these notations,

$$m^*(\Delta_L - E)m \;=\; f^*(\Delta_L - E)f \;=\; \mu \;.$$

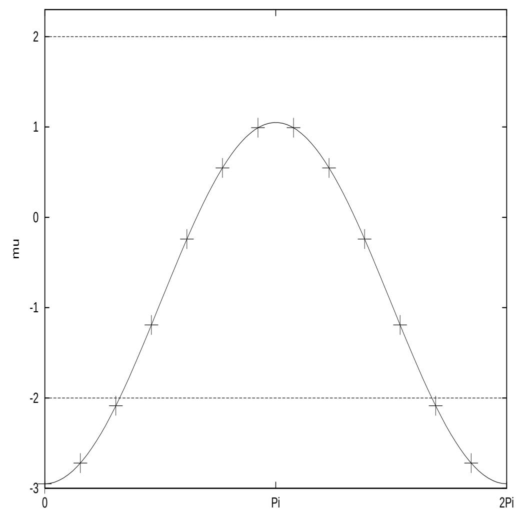


Figure 1: *Plot of the the energy levels for* $L = 13$. (i) *Here* $E = 0.95$ *and* $L_h = 2$. *The eigenvalues* $\mu_0$, $\mu_1$, $\mu_2$, $\mu_{11}$ *and* $\mu_{12}$ *are outside of the window* $[-2, 2]$ *and hence hyperbolic.* (ii) *Here* $E = -0.03$. *All eigenvalues are elliptic.*

An eigenvalue $\mu_l$ will be called elliptic if $|\mu_l| < 2$, hyperbolic if $|\mu_l| > 2$ and parabolic if $|\mu_l| = 2$. Here energies $E$ for which there are parabolic eigenvalues are excluded by the hypothesis (3). An energy $E$ is in the spectrum of $H_L(0)$, the Laplacian on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ if there exists an elliptic or a parabolic eigenvalue. The spectrum of $H_L(0)$ is hence $[-4, 4]$ if $L > 2$ is even and $[-4, 2 + 2\cos(\pi/L)]$ if $L > 1$ is odd. If $L$ is odd and $E$ slightly above the band center (Fig. 1(ii)), all eigenvalues are elliptic while for energies $E$ outside of the spectrum, all eigenvalues are hyperbolic. In between one has both hyperbolic and elliptic eigenvalues. For notational convenience, we will suppose that $\mu_l < 2$ for all $l$. Hence there exists $L_h \leq \frac{L}{2}$ such that that $\mu_l$ is hyperbolic for $l = 0, \ldots, L_h$, and elliptic for $l = L_h + 1, \ldots, [\frac{L}{2}]$. Moreover, there is a degeneracy $\mu_{L-l} = \mu_l$ due to reflection symmetry which will be further analyzed in Section 2.3. In case there are no hyperbolic eigenvalues, let us set $L_h = -1$. In Fig. 1 are shown examples of (i) a situation with mixed elliptic and hyperbolic eigenvalues and (ii) a situation with only elliptic ones.

For later use, let us set $g = \mathrm{diag}(g_1, \ldots, g_L)$ where $g_l = 1$ if $\mu_l$ is elliptic and $g_l = \imath$ if $\mu_l$ is hyperbolic. Note that $[d, \mu] = [d, g] = [\mu, g] = 0$.

In order to diagonalize the transfer matrix, let us introduce the diagonal complex $L \times L$ matrix $\kappa = \frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} - 1}$ as well as the (possibly complex-valued) matrices

$$N = \begin{pmatrix} m\sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}} & m\sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}} \\ m\sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}}\,\frac{1}{\kappa} & m\sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}}\,\kappa \end{pmatrix}, \qquad N^{-1} = \begin{pmatrix} \sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}}\,\kappa\, m^* & -\sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}}\, m^* \\ -\sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}}\,\frac{1}{\kappa}\, m^* & \sqrt{\frac{1}{\kappa - \frac{1}{\kappa}}}\, m^* \end{pmatrix},$$

where here and below all roots are taken on the first branch. Then one immediately verifies that

$$N^{-1}\begin{pmatrix}\Delta_L - E & -\mathbf{1}\\ \mathbf{1} & 0\end{pmatrix} N \;=\; \begin{pmatrix}\kappa & 0\\ 0 & \frac{1}{\kappa}\end{pmatrix} .$$

Finally the r.h.s. will be transformed into a normal form which is real-valued and symplectic. For elliptic $\mu_l$ define $0<\eta_l<\pi$ by $\kappa_l = e^{\imath\eta_l}$, for a hyperbolic one $\eta_l>0$ by $\kappa_l = e^{\eta_l}$. Then introduce the rotation and hyperbolic rotation matrices by ($\eta\in\mathbb{R}$):

$$R_e(\eta) \;=\; \begin{pmatrix}\cos(\eta) & -\sin(\eta)\\ \sin(\eta) & \cos(\eta)\end{pmatrix} , \qquad R_h(\eta) \;=\; \begin{pmatrix}\cosh(\eta) & \sinh(\eta)\\ \sinh(\eta) & \cosh(\eta)\end{pmatrix} .$$

Setting

$$C \;=\; \sqrt{\frac{\imath}{2}}\begin{pmatrix}\mathbf{1} & \imath\,\mathbf{1}\\ \mathbf{1} & -\imath\,\mathbf{1}\end{pmatrix} , \qquad G \;=\; \begin{pmatrix}\overline{g^{\frac{1}{2}}} & 0\\ 0 & g^2\,g^{\frac{1}{2}}\end{pmatrix} ,$$

one verifies

$$G^{-1}\,C^{-1}\begin{pmatrix}\kappa & 0\\ 0 & \frac{1}{\kappa}\end{pmatrix} C\,G \;=\; R_1(\eta_1)\oplus\ldots\oplus R_L(\eta_L) ,$$

where $R_l(\eta_l)$ is either $R_e(\eta_l)$ or $R_h(\eta_l)$ depending on whether $\mu_l$ is elliptic or hyperbolic, and where the direct sum is understood such that $R_l(\eta_l)$ acts on the $l$th and $(l+L)$th component of $\mathbb{C}^{2L}$, namely the normal form $R = R_1(\eta_1)\oplus\ldots\oplus R_L(\eta_L)$ is a real-valued symplectic matrix. In case that there are parabolic eigenvalues, the normal form contains Jordan blocs just as in [SB].

Let us resume the above. Set $M = NCG$, then

$$M^{-1}\begin{pmatrix}\Delta_L - E & -\mathbf{1}\\ \mathbf{1} & 0\end{pmatrix} M \;=\; R .$$

Furthermore the symplectic basis change $M\in\mathrm{SP}(2L,\mathbb{R})$ is more explicitly given by

$$M \;=\; \begin{pmatrix} m\,h & 0\\ m\,\frac{1}{2}\,(\kappa+\frac{1}{\kappa})\,h & m\,h^{-1}\,g^2\end{pmatrix} , \qquad M^{-1} \;=\; \begin{pmatrix} h^{-1}\,m^* & 0\\ -\frac{1}{2}\,g^2\,(\kappa+\frac{1}{\kappa})\,h\,m^* & g^2\,h\,m^*\end{pmatrix} ,$$

where

$$h \;=\; \sqrt{\frac{2\imath\,g^3}{\kappa-\frac{1}{\kappa}}} \;=\; \mathrm{diag}(h_1,\ldots,h_L) , \qquad h_l \;=\; \begin{cases} \sin(\eta_l)^{-\frac{1}{2}} & \mu_l \text{ elliptic,}\\ \sinh(\eta_l)^{-\frac{1}{2}} & \mu_l \text{ hyperbolic.}\end{cases}$$

## 2.2 Calculation of the perturbation

First note that

$$T \;=\; \begin{pmatrix} \Delta_L - E & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \left( \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} - \lambda \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \right) .$$

Hence $P \in \mathrm{sp}(2L,\mathbb{R})$ in equation (5) is given by

$$P \;=\; M^{-1} \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} M \;=\; \begin{pmatrix} 0 & 0 \\ g^2\, h\, d^*\, \hat{V}\, d\, h & 0 \end{pmatrix} . \tag{6}$$

Here the identity $m = fd$ was used as well as the definition $\hat{V} = f^* V f$. Actually $f$ is the matrix of the discrete Fourier transform so that $\hat{V} = \hat{V}^*$ is the Toeplitz matrix associated with the Fourier transform of the random potential (at fixed height $n$). More precisely, define

$$\hat{v}(k) \;=\; \frac{1}{L} \sum_{l=1}^{L} v(l)\, \exp\left( \frac{2\pi\imath\, lk}{L} \right) .$$

One has $\overline{\hat{v}(k)} = \hat{v}(-k)$ and $\hat{v}(k+L) = \hat{v}(k)$. Then

$$\hat{V} \;=\; \begin{pmatrix} \hat{v}(0) & \hat{v}(1) & \dots & \hat{v}(L-1) \\ \hat{v}(-1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{v}(1) \\ \hat{v}(-L+1) & \dots & \hat{v}(-1) & \hat{v}(0) \end{pmatrix} .$$

## 2.3 Symplectic channels

The symplectic channels of a symplectic matrix are by definition the eigenspaces of its (possibly degenerate) eigenvalue pairs $(\kappa, \frac{1}{\kappa})$. For the elliptic-hyperbolic rotation $R = R_1(\eta_1) \oplus \ldots \oplus R_L(\eta_L)$, they are the maximal subspaces of $\mathbb{C}^{2L}$ characterized by the property that $R$ rotates by the same angle. For the Anderson model on a strip studied here, there are $L_c + 1 = [\frac{L}{2}] + 1$ channels. The 0th channel is associated with the fundamental $\mu_L$ and given by the span of $\{e_L, e_{2L}\}$ where $e_l \in \mathbb{C}^{2L}$ has a non-vanshing entry equal to 1 only in the $l$th component. If $L$ is even, also the $\frac{L}{2}$th channel is simple and spanned by $\{e_{L/2}, e_{3L/2}\}$. Due to the degeneracy $\mu_l = \mu_{L-l}$ of the spectrum of $\Delta_L$, all other channels are doubly degenerate. For $l \neq 0, \frac{L}{2}$, the $l$th channel is spanned by $\{e_l, e_{L-l}, e_{l+L}, e_{2L-l}\}$. Let us denote the degeneracy of the $l$th channel by $\nu_l$. In accordance with the above, the $l$th channel is called elliptic if $|\mu_l| < 2$ and hyperbolic if $|\mu_l| > 2$. With these notations, $\mathbb{C}^{2L}$ is hence decomposed into a direct sum of the $L_c + 1$ channels.

There are $L_h + 1$ hyperbolic channels. As we supposed (for convenience) that $\mu_l < 2$, the hyperbolic channels are the first ones, namely channels $0, \ldots, L_h$ (compare Fig. 1(i)).

The projection on the $l$th channel will be denoted by $\pi_l$. It satisfies $[\pi_l, J] = 0$. The non-zero eigenvalues of $\pi_l R \pi_l$ are $e^{\pm \imath \eta_l}$ if the $l$th channel is elliptic and $e^{\pm \eta_l}$ if it is hyperbolic. Hence one can decompose $\pi_l$ into the corresponding eigenspaces $\pi_l = \pi_l^+ + \pi_l^-$. Resuming all the above,

$$\mathbf{1}_{\mathbb{C}^{2L}} \;=\; \sum_{l=0}^{L_c} \pi_l^+ + \pi_l^- \;, \qquad R\,\pi_l^{\pm} \;=\; e^{\pm\,\imath\,\overline{g_l}\,\eta_l}\;\pi_l^{\pm}\;. \tag{7}$$

## 2.4 A privileged basis

For explicit calculations in the next section, it will be convenient to dispose of a basis of $\mathbb{C}^{2L}$ with the following properties:

1. the basis vectors are orthonormal and compatible with the symplectic structure;
2. the basis vectors are eigenvectors of $R$;
3. the matrix elements of the perturbation $P$ w.r.t. the basis are particularly simple, *i.e.*, up to a constant, given by the Fourier transform $\hat{v}$ of the potential.

For this purpose let us note that, for any $\eta \in \mathbb{R}$, linearly independent eigenvectors of $R_e(\eta)$ are $\begin{pmatrix} 1 \\ -\imath \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \imath \end{pmatrix}$, while for $R_h(\eta)$ one can choose $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Comparing with equation (6), the third property is verified if the top $L$ components of the basis vectors commute with $h$ and are constructed with the inverse $d^*$ of $d$ or its quasi-inverse $d^t$. Indeed it can be verified that $dd^t = \mathcal{S}$ is the reflection in $\mathbb{C}^L$ sending component $l$ to component $(L-l)$. In particular, the components $\frac{L}{2}$ and $L$ are left invariant by $\mathcal{S}$. Moreover, $\mathcal{S}$ preserves the channels, namely $[\mathcal{S}, d] = [\mathcal{S}, g] = [\mathcal{S}, h] = 0$ and $\begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{pmatrix}$ commutes with $\pi_l$. Hence one is led to define the following:

$$W \;=\; \left(\mathrm{w}_1^+, \ldots, \mathrm{w}_L^+, \mathrm{w}_1^-, \ldots, \mathrm{w}_L^-\right) \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} d^* & d^t \\ -\imath g\, d^* & \imath g\, d^t \end{pmatrix} \;. \tag{8}$$

It can readily be verified that

$$W^* W \;=\; \mathbf{1}\;, \qquad W^*\, J\, W \;=\; \frac{\imath}{2} \begin{pmatrix} g + g^* & (g^* - g)\mathcal{S} \\ (g - g^*)\mathcal{S} & -g - g^* \end{pmatrix} \;. \tag{9}$$

Hence the vectors $|\mathrm{w}_1^+\rangle, \ldots, |\mathrm{w}_L^+\rangle, |\mathrm{w}_1^-\rangle, \ldots, |\mathrm{w}_L^-\rangle$ defined by (8) form an orthonormal basis of $\mathbb{C}^{2L}$ (we switch to Dirac notation for vectors here). Let us write out examples for a simple and a double eigenvalue ($\mu_L$ and $\mu_l$, $1 \leq l < \frac{L}{2}$, respectively) more explicitly:

$$|\mathrm{w}_L^+\rangle \;=\; \frac{1}{\sqrt{2}} \left(|e_L\rangle - \imath g_L\, |e_{2L}\rangle\right)\;, \qquad |\mathrm{w}_l^+\rangle \;=\; \frac{1}{2} \left(\imath\, |e_l\rangle + |e_{L-l}\rangle + g_l\, |e_{L+l}\rangle - \imath g_l\, |e_{2L-l}\rangle\right)\;.$$

Note that for an elliptic eigenvalue $\mu_l$, $|\mathrm{w}_l^-\rangle$ is the complex conjugate of $|\mathrm{w}_l^+\rangle$, but if $\mu_l$ is hyperbolic, this only holds for the first $L$ components. As requested, one reads off from (9) that

$$J\,|\mathrm{w}_l^\pm\rangle \;=\; \begin{cases} \pm\,\imath|\mathrm{w}_l^\pm\rangle & \mu_l \;\text{ elliptic },\\ \\ \mp\,|\mathrm{w}_{L-l}^\mp\rangle & \mu_l\;\text{ hyperbolic }. \end{cases} \tag{10}$$

Also the second required property follows from (8):

$$R\,|\mathrm{w}_l^\pm\rangle \;=\; e^{\pm\imath\,\overline{g_l}\,\eta_l}\,|\mathrm{w}_l^\pm\rangle\;.$$

Finally,

$$\pi_0^\pm \;=\; |\mathrm{w}_L^\pm\rangle\langle\mathrm{w}_L^\pm|\;,\qquad \pi_l^\pm \;=\; |\mathrm{w}_l^\pm\rangle\langle\mathrm{w}_l^\pm| + |\mathrm{w}_{L-l}^\pm\rangle\langle\mathrm{w}_{L-l}^\pm|\;,\qquad \pi_{\frac{L}{2}}^\pm \;=\; |\mathrm{w}_{\frac{L}{2}}^\pm\rangle\langle\mathrm{w}_{\frac{L}{2}}^\pm|\;. \tag{11}$$

## 2.5 Matrix elements of the random perturbation

In this section, we show that the third desired property of the basis stated in Section 2.4 is fulfilled and then exploit it in order to calculate the matrix elements of the perturbation and some expectation values thereof. Taking into account that $[d,h]=0$, it now follows from (6) and (8) that

$$W^*PW \;=\; \frac{\imath}{2}\begin{pmatrix} gh\hat{V}h & gh\hat{V}h\mathcal{S} \\ -\mathcal{S}gh\hat{V}h & -\mathcal{S}gh\hat{V}h\mathcal{S}\end{pmatrix}\;,\qquad W^*P^*W \;=\; \frac{\imath}{2}\begin{pmatrix} -h\hat{V}hg^* & h\hat{V}hg^*\mathcal{S} \\ -\mathcal{S}h\hat{V}hg^* & \mathcal{S}h\hat{V}hg^*\mathcal{S}\end{pmatrix}\;. \tag{12}$$

Hence one can read off, for two signs $\tau,\sigma$,

$$\langle\mathrm{w}_l^\tau|P|\mathrm{w}_k^\sigma\rangle \;=\; \tau\,\frac{\imath}{2}\,g_l\,h_lh_k\,\hat{v}(\sigma k-\tau l)\;,\qquad \langle\mathrm{w}_l^\tau|P^*|\mathrm{w}_k^\sigma\rangle \;=\; -\sigma\,\frac{\imath}{2}\,\overline{g_k}\,h_lh_k\,\hat{v}(\sigma k-\tau l)\;. \tag{13}$$

Let us collect some useful identities.

**Lemma 1** *Let $w_j^\pm\in\mathbb{C}^{2L}$ be unit vectors satisfying $\pi_j^\pm w_j^\pm = w_j^\pm$. Let $\tilde{P}=P+P^*$.*
(i) $\mathbf{E}\,\langle w|P|w'\rangle = 0$ *for any $w,w'\in\mathbb{C}^{2L}$.*
(ii) *If either $k\neq l$ or $\sigma\neq\tau$, then*

$$\mathbf{E}\,\langle w_l^\tau|\,P\,|w_k^\sigma\rangle\;\langle w_k^\sigma|\,P\,|w_l^\tau\rangle \;=\; -\tau\sigma\,\frac{1}{4L}\,g_l\,g_k\,h_l^2h_k^2\;.$$

(iii) *If either $k\neq l$ or $\sigma\neq\tau$, then*

$$\mathbf{E}\,\langle w_l^\tau|\,P^*\,|w_k^\sigma\rangle\;\langle w_k^\sigma|\,P\,|w_l^\tau\rangle \;=\; \frac{1}{4L}\,h_l^2h_k^2\;.$$

(iv) *Let channels $l,k$ be elliptic. Then $\pi_l^\sigma\tilde{P}\pi_k^\sigma=0$ and $\pi_l^\sigma\tilde{P}\pi_k^{-\sigma} = 2\,\pi_l^\sigma P\pi_k^{-\sigma}$. Moreover,*

$$\mathbf{E}\,\langle w_l^\sigma|\,\tilde{P}\,|w_k^{-\sigma}\rangle\;\langle w_k^{-\sigma}|\,\tilde{P}\,|w_l^\sigma\rangle \;=\; \frac{1}{L}\,h_l^2 h_k^2 \;=\; \mathbf{E}\,\langle w_l^\sigma|\,\tilde{P}\,|w_k^{-\sigma}\rangle\;\langle\overline{w_l^\sigma}|\,\tilde{P}\,|\overline{w_k^{-\sigma}}\rangle\;.$$

(v)

$$\mathbf{E}\,\langle w_l^\sigma|\,|RP|^2\,|w_l^\sigma\rangle \;=\; \frac{1}{2}\,h_{\mathrm{av}}^2\,h_l^2\;, \qquad \textit{where} \qquad h_{\mathrm{av}}^2 \;=\; \frac{1}{L}\sum_{k=0}^{L_c}\nu_k\,h_k^2\,\cosh((1-g_k^2)\eta_k)\;.$$

One might think of items (ii) through (v) as follows. Even though the perturbation $P$ lifts the degeneracy of the channels, taking expectation values re-establishes it.

**Proof.** (i) This follows directly from $\mathbf{E}\,v(l)=0$ for all $l$.

(ii) Let $w_l^\tau = a_l \mathrm{w}_l^\tau + b_l \mathrm{w}_{L-l}^\tau$ with $|a_l|^2+|b_l|^2=1$ and $b_l=0$ if the channel is simple, that is $l=0,\frac{L}{2}$. With these notations, it follows from (13)

$$\begin{aligned}\langle w_l^\tau|\,P\,|w_k^\sigma\rangle\;\langle w_k^\sigma|\,P\,|w_l^\tau\rangle \;&=\; -\sigma\tau\,\frac{1}{4}\,g_l\,g_k\,h_l^2h_k^2\;\cdot\\ &\cdot\big(\overline{a_l}a_k\hat{v}(\sigma k-\tau l)+\overline{a_l}b_k\hat{v}(-\sigma k-\tau l)+\overline{b_l}a_k\hat{v}(\sigma k+\tau l)+\overline{b_l}b_k\hat{v}(-\sigma k+\tau l)\big)\;\cdot\\ &\cdot\big(a_l\overline{a_k}\hat{v}(-\sigma k+\tau l)+a_l\overline{b_k}\hat{v}(-\sigma k-\tau l)+b_l\overline{a_k}\hat{v}(\sigma k+\tau l)+b_l\overline{b_k}\hat{v}(\sigma k-\tau l)\big)\;.\end{aligned}$$

Now $\mathbf{E}\,v(l)^2=1$ implies that $\mathbf{E}\,\hat{v}(q)\hat{v}(p)=\frac{1}{L}\,\delta_{q,-p}$. With a bit of care, one can now check that the expectation value of the product of the last two factors is equal to 1.

(iii) This is proven in the same manner as (ii).

(iv) As $g_l=g_k=1$, it follows from (12) that $W^*\tilde{P}W=W^*(P+P^*)W$ has only off-diagonal entries in $(\pi_l+\pi_k)\mathbb{C}^{2L}$ equal to twice those of $W^*PW$. The first equality now follows directly from (ii), while the second one is checked similarly.

(v) One first verifies as above that (also for $l=k$ and $\sigma=\pm$)

$$\mathbf{E}\,\langle w_l^\sigma|P^*|\mathrm{w}_k^\pm\rangle\langle \mathrm{w}_k^\pm|P|w_l^\sigma\rangle \;=\; \frac{1}{4L}\,h_k^2\,h_l^2\;.$$

Hence the claim follows by inserting an identity (7) and summing over $k$. □

# 3 Random dynamics of symplectic frames

## 3.1 Symplectic frames and isotropic manifolds

The space $\mathcal{F}_p$ of symplectic $p$-frames, $p=1,\ldots,L$, is defined by

$$\mathcal{F}_p \;=\; \big\{(u_1,\ldots,u_p)\;\big|\;u_l\in\mathbb{R}^{2L}\;,\;\langle u_l|u_k\rangle=\delta_{l,k}\;,\;\langle u_l|J|u_k\rangle=0\;,\;l,k=1,\ldots,p\;\big\}\;.$$

It is a manifold of dimension $p(2L-p)$. One could also call $\mathcal{F}_p$ an isotropic Stiefel manifold and $\mathcal{F}_L$ the Langrangian Stiefel manifold.

**Proposition 1** *The map* $\zeta : \mathcal{F}_L \to \mathrm{O}(2L,\mathbb{R}) \cap \mathrm{SP}(2L,\mathbb{R}) \cong \mathrm{U}(L,\mathbb{C})$ *defined by*

$$\zeta(u) \;=\; (u, Ju) \;, \qquad u \;=\; (u_1,\ldots,u_L) \in \mathcal{F}_L \;,$$

*is an isomorphism.*

**Proof.** This is immediate if one recalls

$$\mathrm{O}(2L,\mathbb{R}) \cap \mathrm{SP}(2L,\mathbb{R}) \;=\; \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| \, a, b \in \mathrm{M}_{L\times L}(\mathbb{R}) \;,\; a^t a + b^t b = \mathbf{1} \;,\; a^t b = b^t a \right\} \;.$$

Moreover, $a + \imath\, b \in \mathrm{U}(L,\mathbb{C})$ gives the second isomorphism. □

A subspace $\mathcal{E} \subset \mathbb{R}^{2L}$ is called symplectic if $\langle v|J|v'\rangle = 0$ for all $v, v' \in \mathcal{E}$. The isotropic manifold $L_p$ is by definition the set of all oriented symplectic $p$-dimensional planes in $\mathbb{R}^{2L}$. It is a manifold of dimension $2p(L-p) + \frac{1}{2}\,p(p+1)$. The maximal isotropic manifold $L_L$ is also called Lagrangian manifold.

Next let $\Lambda^p\mathbb{C}^{2L}$, $p = 1,\ldots,L$, be the vector spaces of the anti-symmetrized $p$-fold tensor products of $\mathbb{C}^{2L}$. Decomposable (unentangled) vectors therein will be denoted by $u_1 \wedge \ldots \wedge u_p$ where $u_l \in \mathbb{C}^{2L}$. A scalar product on $\Lambda^p\mathbb{C}^{2L}$ is defined as usual by

$$\langle u_1 \wedge \ldots \wedge u_p \,|\, u_1' \wedge \ldots \wedge u_p'\rangle_{\Lambda^p\mathbb{C}^{2L}} \;=\; \det{}_p\left(\langle u_l \,|\, u_k'\rangle_{1\leq l,k\leq p}\right) \;.$$

As is well-known, oriented $p$-dimensional planes are isomorphic to the set of decomposable real unit vectors in $\Lambda^p\mathbb{C}^{2L}$. For the isotropic manifolds, this implies

$$L_p \;\cong\; \left\{ v_1 \wedge \ldots \wedge v_p \in \Lambda^p\mathbb{R}^{2L} \,\middle|\, \|v_1 \wedge \ldots \wedge v_p\| = 1 \;,\; \langle v_l|J|v_k\rangle = 0 \;,\; l,k = 1,\ldots,p \right\} \;.$$

Now each element $u = (u_1,\ldots,u_p) \in \mathcal{F}_p$ defines a sequence of embedded, oriented symplectic planes $u_1 \wedge \ldots \wedge u_q \in L_q$, $q = 1,\ldots,p$. Expressed in a different way, $u \in \mathcal{F}_p$ gives an element $u_1 \wedge \ldots \wedge u_p \in L_p$ as well as an unoriented, but ordered orthonormal basis therein. This is locally an isomorphism:

**Proposition 2** $\mathcal{F}_p$ *is a principal bundle over* $L_p$ *with fiber* $\mathrm{SO}(p,\mathbb{R})$.

The following elementary lemma about matrix elements of a Lagrangian projection w.r.t. eigenvectors of $J$ (and hence also all rotations $R$ constructed in Section 2.1) will be used later on.

**Lemma 2** *Let* $\Pi = uu^t$ *be the projection on the Lagrangian plane associated with* $u \in \mathcal{F}_L$. *If* $v_j \in \mathbb{C}^{2L}$, $j = 1,2$, *are two normalized orthogonal eigenvectors of* $J$, *namely* $Jv_j = \imath\sigma_j v_j$ *for signs* $\sigma_j$, *then*

$$(1 + \sigma_j\sigma_k)\;\langle v_j|\Pi|v_k\rangle \;=\; \delta_{j,k} \;.$$

**Proof.** Let $\Pi_1$ and $\Pi_2$ be the projections on the first and second $L$ components of $\mathbb{C}^{2L}$ respectively. Hence $\Pi_1 + \Pi_2 = \mathbf{1}$. Using the orthogonal $(u, Ju)$ of Proposition 1, one then has

$$\delta_{j,k} \;=\; \langle v_j|(u,Ju)(\Pi_1+\Pi_2)(u,Ju)^t|v_k\rangle \;=\; \langle v_j|uu^t|v_k\rangle + \langle v_j|Juu^tJ^t|v_k\rangle \;.$$

As $J^t = J^* = -J$, the claim follows from the supposed properties of $v_j$. □

## 3.2 Action of a transfer matrix on a symplectic frame

The group $\mathrm{SP}(2L,\mathbb{R})$ acts on the space $\mathcal{F}_L$ of symplectic $L$-frames. There is an obvious way to define such an action $\mathcal{U} : \mathrm{SP}(2L,\mathbb{R}) \times \mathcal{F}_L \to \mathcal{F}_L$ (which will later on actually turn out to be relevant for the calculation of the Lyapunov exponents): given $T \in \mathrm{SP}(2L,\mathbb{R})$ and $u = (u_1,\ldots,u_L) \in \mathcal{F}_L$, the plane $Tu_1 \wedge \ldots \wedge Tu_p$ is symplectic for any $p \leq L$; hence applying the Schmidt orthonormalization procedure to the sequence $Tu_1,\ldots,Tu_L$ gives a new element $\mathcal{U}_T u \in \mathcal{F}_L$. On a calculatory level, it will be convenient (and equivalent as one easily verifies) to define this action using wedge products:

$$\mathcal{U}_T\, u_1 \wedge \ldots \wedge u_p \;=\; \frac{\Lambda^p T\, u_1 \wedge \ldots \wedge u_p}{\|\Lambda^p T\, u_1 \wedge \ldots \wedge u_p\|}\;, \qquad p = 1,\ldots,L\;. \tag{14}$$

More explicitly, this means that the $p$th vector of the new frame $(\mathcal{U}_T\, u)_p$ satisfies for all $v \in \mathbb{C}^{2L}$:

$$\langle v|(\mathcal{U}_T\, u)_p\rangle \;=\; \frac{\langle (\Lambda^{p-1}T\, u_1 \wedge \ldots \wedge u_{p-1}) \wedge v|\Lambda^p T\, u_1 \wedge \ldots \wedge u_p\rangle}{\|\Lambda^{p-1}T\, u_1 \wedge \ldots \wedge u_{p-1}\|\;\|\Lambda^p T\, u_1 \wedge \ldots \wedge u_p\|}\;. \tag{15}$$

It is immediate from the definition that

$$\mathcal{U}_{ST} \;=\; \mathcal{U}_S\,\mathcal{U}_T\;, \qquad S,T \in\; \mathrm{SP}(2L,\mathbb{R})\;.$$

For fixed $p$, (14) defines a map $\mathcal{U}_T$ on $L_p$. But the whole sequence $p = 1,\ldots,L$, defines a map on $\mathcal{F}_L$. Due to Proposition 1, this defines an action on the unitary group $\mathrm{U}(L,\mathbb{C})$. Therefore each $\mathcal{U}_T$ can be identified with an element of $\mathrm{U}(L,\mathbb{C})$ itself (explaining hence the notation with a letter $\mathcal{U}$).

## 3.3 Definition of random dynamics

The random transfer matrices are symplectic so that they induce an action on the frames. Here their transformed normal form $R(1-\lambda P(n))$ given by (5) will be used to define a random dynamical system on $\mathcal{F}_L$. The random orbits in $\mathcal{F}_L$ for some fixed initial condition $(u_1(0),\ldots,u_L(0))$ will be defined and denoted as follows:

$$u_1(n) \wedge \ldots \wedge u_L(n) \;=\; \mathcal{U}_{R(1-\lambda P(n))} u_1(n-1) \wedge \ldots \wedge u_L(n-1)\;, \qquad n \geq 1\;. \tag{16}$$

The free dynamics at $\lambda = 0$ is non-random and just given by the (elliptic-hyperbolic) rotation $R$, the analysis of which is straight-forward. The main objects of this work is to study the effect of weakly coupled randomness.

All the information needed for calculating averaged quantities like the Lyapunov exponent can actually be encoded in an invariant measure $\nu$ on $\mathcal{F}_L$. It can be defined by

$$\int_{\mathcal{F}_L} d\nu(u)\, f(u) \;=\; \mathbf{E} \int_{\mathcal{F}_L} d\nu(u)\, f(\mathcal{U}_{R(1-\lambda P)}u) \;.$$

It is known [BL] that $\nu$ is unique as soon as $\lambda \neq 0$ and moreover continuous. In [JSS, Section 4.7], perturbation theory for this measure (around the Lebesgue measure) was done in the case $L = 1$. The results below (in particular, Section 4.3) can be interpreted in a similar way.

## 3.4 Channel weights

It will be useful to introduce the probability (or weight) $\rho_{p,j}(n) \in [0, 1]$ for the $p$th frame vector $u_p(n)$ to be in the $j$th channel. More explicitly,

$$\rho^{\pm}_{p,j}(n) \;=\; \langle u_p(n)|\pi^{\pm}_j|u_p(n)\rangle \;, \qquad \rho_{p,j}(n) \;=\; \rho^{+}_{p,j}(n) + \rho^{-}_{p,j}(n) \;. \tag{17}$$

The identity (7) and Lemma 2 (use $2\nu_j$ eigenvectors of $J$ to build a basis in $\pi_j\mathbb{C}^{2L}$) imply respectively

$$\sum_{j=0}^{L_c} \rho_{p,j}(n) \;=\; 1 \;, \qquad \sum_{p=1}^{L} \rho_{p,j}(n) \;=\; \nu_j \;. \tag{18}$$

For an elliptic channel one has $\pi^{-}_j = \overline{\pi^{+}_j}$ so that

$$\rho_{p,j}(n) \;=\; 2\,\rho^{\pm}_{p,j}(n) \;, \qquad \mu_j \;\; \text{elliptic} \;. \tag{19}$$

For the random dynamics of frames, let us introduce the mean presence probability of the $p$th frame vector in the $j$th channel:

$$\langle \rho_{p,j}\rangle_N \;=\; \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\rho_{p,j}(n) \;, \qquad \langle \rho_{p,j}\rho_{q,k}\rangle_N \;=\; \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\rho_{p,j}(n)\rho_{q,k}(n) \;. \tag{20}$$

Similarly, higher moments are defined. It follows from a standard ergodic argument that these quantities converge in the limit $N \to \infty$ to some numbers denoted $\langle \rho_{p,j}\rangle$, $\langle \rho_{p,j}\rho_{q,k}\rangle$ and so on. However, this fact will not be used below.

## 3.5 Separating hyperbolic from elliptic channels

A bit of thought shows that the first $2L_h+1$ frame vectors $u_1, \ldots, u_{2L_h+1}$ deterministically aline (up to an error) with the expanding hyperbolic basis vectors, that is $\mathrm{w}^{+}_1, \ldots, \mathrm{w}^{+}_{L_h}, \mathrm{w}^{+}_{L-L_h}, \ldots, \mathrm{w}^{+}_L$ (recall that we supposed $\mu_l < 2$, as in Fig. 1(i)). Hence the remaining frame vectors have to be in the elliptic channels due to orthogonal and symplectic blocking. Hence let us call the frame vectors $u_{2L_h+2}, \ldots, u_L$ elliptic, while the first ones are called hyperbolic. The corresponding analysis, elementary but a bit tedious, is carried out below. It is not needed in case all channels are elliptic (as in Fig. 1(ii)).

**Proposition 3** *For almost every initial condition (or almost every disorder configuration), for $n$ sufficiently large and for $k = 1, \ldots, L_h$,*

$$|\langle \mathrm{w}_L^+ | u_1(n)\rangle|^2 \;=\; 1 - \mathcal{O}(\lambda^2) \;, \qquad |\langle \mathrm{w}_k^+ \wedge \mathrm{w}_{L-k}^+ | u_{2k}(n) \wedge u_{2k+1}(n)\rangle|^2 \;=\; 1 - \mathcal{O}(\lambda^2) \;.$$

**Proof.** Let us begin with the study of $|\langle \mathrm{w}_L^+ | u_1(n)\rangle|^2$. It follows from (14) and (16) that

$$|\langle \mathrm{w}_L^+ | u_1(n+1)\rangle|^2 \;=\; \frac{e^{2\eta_L}\, |\langle \mathrm{w}_L^+ | u_1(n)\rangle|^2}{\langle u_1(n)|R^*R|u_1(n)\rangle} \;+\; \mathcal{O}(\lambda) \;. \tag{21}$$

In order to analyse the denominator, let us insert (7):

$$\begin{aligned}
\langle u_1(n)|R^*R|u_1(n)\rangle \;&=\; \sum_{l=1}^{L} e^{\eta_l(1-g_l^2)}|\langle \mathrm{w}_l^+|u_1(n)\rangle|^2 + e^{-\eta_l(1-g_l^2)}|\langle \mathrm{w}_l^-|u_1(n)\rangle|^2 \\
&=\; 1 + \sum_{l=0}^{L_h} (e^{2\eta_l} - 1)\, |\langle \mathrm{w}_l^+|u_1(n)\rangle|^2 + (e^{-2\eta_l} - 1)|\langle \mathrm{w}_l^-|u_1(n)\rangle|^2 \;,
\end{aligned}$$

where as before we identified indices $L \widehat{=} 0$ and used that $u_1(n)$ is normalized. The next aim is to prove an upper bound on this and therefore let us first note that $e^{-2\eta_l} - 1 < 0$ so that those terms can be discarded. For the remainder, starting from the smallest factor $e^{2\eta_l} - 1$ and going iteratively to the largest, one gets using each time $|\langle \mathrm{w}_l^+|u_1(n)\rangle|^2 \leq 1 - \sum_{k=0}^{l-1} |\langle \mathrm{w}_k^+|u_1(n)\rangle|^2$,

$$\begin{aligned}
\langle u_1(n)|R^*R|u_1(n)\rangle \;&\leq\; e^{2\eta_{L_h}} + \sum_{l=0}^{L_h-1} (e^{2\eta_l} - e^{2\eta_{L_h}})\, |\langle \mathrm{w}_l^+|u_1(n)\rangle|^2 \\
&\leq\; e^{2\eta_1} + (e^{2\eta_L} - e^{2\eta_1})\, |\langle \mathrm{w}_L^+|u_1(n)\rangle|^2 \;.
\end{aligned} \tag{22}$$

Note that $\eta_1$ is the second largest hyperbolic angle satisfying $\eta_1 < \eta_L$. Replacing in the above,

$$|\langle \mathrm{w}_L^+ | u_1(n+1)\rangle|^2 \;\geq\; \frac{e^{2(\eta_L-\eta_1)}\, |\langle \mathrm{w}_L^+ | u_1(n)\rangle|^2}{1 + (e^{2(\eta_L-\eta_1)} - 1)\, |\langle \mathrm{w}_L^+ | u_1(n)\rangle|^2} \;+\; \mathcal{O}(\lambda) \;.$$

Thus, up to an error, $|\langle \mathrm{w}_L^+ | u_1(n+1)\rangle|^2$ is larger than the image of $x = |\langle \mathrm{w}_L^+ | u_1(n)\rangle|^2$ under the function $f_a(x) = \frac{ax}{1+(a-1)x}$ where $a = e^{2(\eta_L-\eta_1)} > 1$. As $n$ grows, this procedure is then iterated, giving rise to a discrete time dynamics through successive application of $f_a$. The function $f_a$ has two fixed points in $[0, 1]$, an unstable one at 0 and a stable one at 1. Either the initial condition is already away from the unstable fixed point or the random perturbation leads the discrete time dynamics to leave it (only with exponentially small probability one remains in its neighborhood, as an elementary argument shows). As it only takes a finite number of iterations to get within the neighborhood of the stable fixed point and the (random) perturbation is of order $\mathcal{O}(\lambda)$, one can conclude that, for $n$ large enough,

$$|\langle \mathrm{w}_L^+|u_1(n)\rangle|^2 \;=\; 1-\mathcal{O}(\lambda)\;.$$

When this holds, however, the random perturbation cannot be linear in $\lambda$ anymore, because the $\mathcal{O}(\lambda)$-term would not have definite sign and hence violate $|\langle \mathrm{w}_L^+|u_1(n)\rangle|^2 \leq 1$. Indeed, it is elementary to verify also algebraically that the perturbative terms linear in $\lambda$ vanish in (21) when one already knows $\langle \mathrm{w}_L^+|u_1(n)\rangle = 1-\mathcal{O}(\lambda)$. Hence one can repeat the above argument in the neighborhood of the stable fixed point 1, but based on (21) with an error term $\mathcal{O}(\lambda^2)$. This implies the first claim. Moreover, due to normalization, $|\langle \mathrm{w}_l^+|u_1(n)\rangle|^2 = \mathcal{O}(\lambda^2)$ for all $l \neq L$ as well as $|\langle \mathrm{w}_l^-|u_1(n)\rangle|^2 = \mathcal{O}(\lambda^2)$ for any $l$.

The remaining estimates are proven by recurrence over $k$. After having exploited orthogonal and symplecting blocking, the basic argument is as before and therefore some calculatory details are suppressed and left to the reader. Hence let us suppose that $|\langle \mathrm{w}_l^+\wedge \mathrm{w}_{L-l}^+|u_{2l}(n)\wedge u_{2l+1}(n)\rangle|^2 = 1-\mathcal{O}(\lambda^2)$ for all $l<k$. As $\langle u_{2k}(n)|u_m(n)\rangle = \delta_{2k,m}$ and $\langle u_{2k}(n)|J|u_m(n)\rangle = 0$, one concludes that $\langle u_{2k}(n)|\mathrm{w}_m^+\rangle = \mathcal{O}(\lambda)$ and $\langle u_{2k}(n)|J|\mathrm{w}_m^+\rangle = \mathcal{O}(\lambda)$ for $m=1,\ldots,k-1,L-k+1,\ldots L$. But for these hyperbolic channels $J|\mathrm{w}_m^+\rangle = |\mathrm{w}_m^-\rangle$. As the same holds for $u_{2k+1}(n)$, we can conclude that

$$\langle \mathrm{w}_m^\sigma \wedge \mathrm{w}_l^\tau|u_{2k}(n)\wedge u_{2k+1}(n)\rangle \;=\; \mathcal{O}(\lambda)\;, \qquad l,m=1,\ldots,k-1,L-k+1,\ldots L\;.$$

Using this, a short perturbative calculation starting from (15) shows

$$|\langle \mathrm{w}_k^+\wedge \mathrm{w}_{L-k}^+|u_{2k}(n+1)\wedge u_{2k+1}(n+1)\rangle|^2 \;=\; \frac{e^{4\eta_k}\;|\langle \mathrm{w}_k^+\wedge \mathrm{w}_{L-k}^+|u_{2k}(n)\wedge u_{2k+1}(n)\rangle|^2}{\langle u_{2k}(n)\wedge u_{2k+1}(n)|\Lambda^2 R^* R|u_{2k}(n)\wedge u_{2k+1}(n)\rangle} + \mathcal{O}(\lambda).$$

Generalizing the argument leading to (22), one can bound the denominator from above by

$$e^{4\eta_{k+1}} \;+\; (e^{4\eta_k}-e^{4\eta_{k+1}})\,|\langle \mathrm{w}_k^+\wedge \mathrm{w}_{L-k}^+|u_{2k}(n)\wedge u_{2k+1}(n)\rangle|^2 \;+\mathcal{O}(\lambda^2)\;.$$

As $\eta_k > \eta_{k+1}$, one can use the same function $f_a$ as above with $a = e^{4(\eta_k-\eta_{k+1})}$ and complete the (two-stepped) argument as above. $\square$

# 4 Lyapunov exponents

## 4.1 Calculating Lyapunov exponents with symplectic frames

In the definition (23) of the Lyapunov exponents appears the operator norm. Instead, one may use symplectic planes as initial condition if an averaging over them is done. This is briefly discussed in this section.

Important is the well-known fact [BL] that for any symplectic matrix $T$, its second quantized is most expansive on the isotropic subspaces, namely the norm of the second quantized $\Lambda^p T$

can be calculated by $\|\Lambda^p T\| = \sup_{u\in L_p} \|\Lambda^p T u\| = \sup_{u\in\mathcal{F}_p} \|\Lambda^p T u\|$. Furthermore, the Lyapunov exponents according to [BL, A.III.3.4] the Lyapunov exponents are given by

$$\sum_{l=1}^{p} \gamma_l \;=\; \lim_{N\to\infty} \frac{1}{N}\,\mathbf{E}\,\log\left(\left\| \prod_{n=1}^{N} \Lambda^p T(n)\; u(0) \right\|_{\Lambda^p\mathbb{C}^{2L}}\right)\;, \qquad (23)$$

where $u(0)\in\mathcal{F}_p$ is an arbitrary initial condition. One may average over $u(0)$ w.r.t. to the invariant measure $\nu$ on $\mathcal{F}_L$ and this immediately leads to

$$\sum_{l=1}^{p} \gamma_l \;=\; \int_{\mathcal{F}_p} d\nu(u)\;\mathbf{E}\,\log\left(\,\|\;\Lambda^p T\;u\;\|_{\Lambda^p\mathbb{C}^{2L}}\right)\;,$$

where here the $\mathbf{E}$ is only an average over the single site transfer matrix $T$. Similar formulas can be found in [CL, Section IV.6].

## 4.2 Basic perturbative formula: only symplectic channels

As it is considerably more transparent, let us first perform the perturbative calculation of the Lyapunov exponents in the case where there are no hyperbolic channels (as in Fig. 1(ii)). Hence we assume $R$ to be orthogonal.

Let us insert $\mathbf{1} = \Lambda^p M \Lambda^p M^{-1}$ in between each pair of transfer matrices in (23) (the boundary terms do not change anything as can easily be argued as in [JSS, Section 4.1], for example). Then develop the product therein into a telescopic sum using the definition of the action (14) as well as the definition of the random dynamics of frames. This gives

$$\sum_{l=1}^{p} \gamma_l \;=\; \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\log\left(\left\|\Lambda^p(M^{-1}T(n+1)M)\;u_1(n)\wedge\ldots\wedge u_p(n)\right\|_{\Lambda^p\mathbb{C}^{2L}}\right)\;,$$

where $\mathbf{E}$ contains also an average over the initial condition $u_1(0)\wedge\ldots\wedge u_p(0)$. As $R$ is orthogonal, so is $\Lambda^p R$. Using $\Lambda^p(M^{-1}T(n)M) = \Lambda^p R\;\Lambda^p(1-\lambda P(n))$, one gets writing out the norm explicitly:

$$\sum_{l=1}^{p} \gamma_l \;=\; \frac{1}{2}\,\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\log\left(\det_p\left(\langle(1-\lambda P(n+1))u_l(n)\,|\,(1-\lambda P(n+1))u_k(n)\rangle_{1\leq l,k\leq p}\right)\right)\;.$$

Now $\log\det_p = \mathrm{Tr}_p\log$, so that multiplying out gives:

$$\sum_{l=1}^{p} \gamma_l \;=\; \frac{1}{2}\,\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\mathrm{Tr}_p\log\left(\mathbf{1}_p + \langle u_l(n)\,|(-\lambda(P+P^*)+\lambda^2|P|^2)|\,u_k(n)\rangle_{1\leq l,k\leq p}\right)\;,$$

where the argument $n+1$ of $P=P(n+1)$ was suppressed because they are all independent and identically distributed random variables over each of which can be averaged independently

in each summand. Finally let $\Pi_p(n)$ be the projection in $\mathbb{R}^{2L}$ onto the subspace spanned by $u_1(n), \ldots, u_p(n)$. Expanding the logarithm up to order $\mathcal{O}(\lambda^3)$ and using that $\mathbf{E}\,\mathrm{Tr}(\Pi_p(n)P) = 0$, we obtain

$$\sum_{l=1}^{p} \gamma_l \;=\; \frac{\lambda^2}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left(\mathrm{Tr}(\Pi_p(n)|P|^2) - \frac{1}{2}\,\mathrm{Tr}(\tilde{P}\Pi_p(n)\tilde{P}\Pi_p(n))\right) + \mathcal{O}(\lambda^3)\;, \tag{24}$$

where the trace is now over $\mathbb{R}^{2L}$ and $\tilde{P} = P + P^*$ is a real and self-adjoint matrix. Subtracting gives, up to $\mathcal{O}(\lambda^3)$,

$$\gamma_p = \frac{\lambda^2}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\left(\langle u_p(n)||P|^2|u_p(n)\rangle - \langle u_p(n)|\tilde{P}\Pi_p(n)\tilde{P}|u_p(n)\rangle + \frac{1}{2}\langle u_p(n)|\tilde{P}|u_p(n)\rangle^2\right). \tag{25}$$

As we shall show in the next section, the first and second contribution cancel exactly for the bottom Lyapunov exponent, while the third one can be calculated explicitly, namely it follows directly from Lemma 3 below (set $L_h = -1$ therein so that there are no hyperbolic channels) that

$$\gamma_L \;=\; \lim_{N\to\infty} \frac{\lambda^2}{8L} \sum_{j,k=0}^{L_c} h_j^2 h_k^2 (2 - \delta_{j,k})\ \langle \rho_{L,j}\rho_{L,k}\rangle_N + \mathcal{O}(\lambda^3)\,. \tag{26}$$

Because $h_j^2 \geq 1$, Theorem 1 follows immediately in the case where there are only elliptic channels.

## 4.3 Oscillatory sums

The aim of this section is to evaluate the terms appearing in the perturbative expansion (25) of the Lyapunov exponent. This will then give (26). As it does not take more effort at this point and will be needed below, we will however not suppose all channels to be elliptic, but only the exterior frame vector $u_p$ to be elliptic, *i.e.* $p > 2L_h + 1$.

**Lemma 3** *Let $p > 2L_h + 1$ and suppose that the Main hypothesis holds.*

(i)

$$\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\,\langle u_p(n)|\,|RP|^2\,|u_p(n)\rangle \;=\; \frac{1}{2}\,h_{\mathrm{av}}^2 \sum_{l=0}^{L_c} h_l^2\ \langle \rho_{p,l}\rangle_N \;+\; \mathcal{O}(N^{-1},\lambda)\;.$$

(ii)

$$\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\,\langle u_p(n)|\tilde{P}|u_p(n)\rangle^2 \;=\; \frac{1}{2L} \sum_{j,k>L_h}^{L_c} h_j^2 h_k^2 (2 - \delta_{j,k})\ \langle \rho_{p,j}\rho_{p,k}\rangle_N \;+\; \mathcal{O}(N^{-1},\lambda)\;.$$

(iii)

$$\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{E}\sum_{q=2L_h+2}^{L}\langle u_p(n)|\tilde{P}|u_q(n)\rangle^2 \;=\; \frac{1}{2}\left(\frac{1}{L}\sum_{l>L_h}^{L_c}\nu_l h_l^2\right)\sum_{k=0}^{L_c}h_k^2\,\langle\rho_{p,k}\rangle_N \;+\;\mathcal{O}(N^{-1},\lambda)\;.$$

**Proof.** (i) By inserting identities (7),

$$\langle u_p(n)|\,|RP|^2\,|u_p(n)\rangle \;=\; \sum_{l,k=0}^{L_c}\langle u_p(n)|(\pi_l^+ + \pi_l^-)\,|RP|^2\,(\pi_k^+ + \pi_k^-)|u_p(n)\rangle\;. \tag{27}$$

By Proposition 3, the sum may be restricted to $L_h+1\leq l,k\leq L_c$ at the cost of an error $\mathcal{O}(\lambda)$. Hence let us consider, for fixed elliptic channels $l,k$, and signs $\sigma,\tau$,

$$J(N) \;=\; \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{E}\,\langle u_p(n)|\pi_l^\sigma\,|RP|^2\,\pi_k^\tau|u_p(n)\rangle\;.$$

As from (14),

$$\pi_k^\tau|u_p(n)\rangle \;=\; e^{\tau\imath\eta_k}\pi_k^\tau|u_p(n-1)\rangle \;+\;\mathcal{O}(\lambda)\;,$$

we get, because the boundary terms are of $\mathcal{O}(N^{-1})$,

$$J(N) \;=\; e^{\imath(-\sigma\eta_l+\tau\eta_k)}\,J(N) \;+\;\mathcal{O}(N^{-1},\lambda)\;.$$

If now $e^{\imath(-\sigma\eta_l+\tau\eta_k)}\neq 1$, this implies $J(N)=\mathcal{O}(N^{-1},\lambda)$. By the main hypothesis this does not happen if $l\neq k$ or $\sigma\neq\mu$. Therefore only the diagonal terms in (27) contribute to leading order so that

$$\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{E}\,\langle u_p(n)|\,|RP|^2\,|u_p(n)\rangle \;=\; \sum_{l=L_h+1}^{L_c}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{\sigma=\pm}\mathbf{E}\,\langle u_p(n)|\pi_l^\sigma|RP|^2\pi_l^\sigma|u_p(n)\rangle \;+\;\mathcal{O}(N^{-1},\lambda)\,.$$

Finally $\pi_l^\sigma|u_p(n)\rangle=(\frac{1}{2}\,\rho_{p,l}(n))^{\frac{1}{2}}\,|w_l^\sigma\rangle$ for some complex unit vector $w_l^\sigma$ satisfying $\pi_l^\sigma|w_l^\sigma\rangle=|w_l^\sigma\rangle$. Thus

$$\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{E}\,\langle u_p(n)|\,|RP|^2\,|u_p(n)\rangle \;=\; \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=L_h+1}^{L_c}\sum_{\sigma=\pm}\mathbf{E}\,\frac{1}{2}\,\rho_{p,l}(n)\,\langle w_l^\sigma|\,|RP|^2\,|w_l^\sigma\rangle+\mathcal{O}(N^{-1},\lambda)\,.$$

But the expectation value of the matrix element (over the random variable $P$ only) is independent of $w_l^\sigma$ and given by Lemma 1(v). This directly leads to the first claim because the sum can again be extended to $l=0,\ldots,L_c$ by Proposition 3.

(ii) One has for $p,q>2L_h+1$

$$\langle u_p(n)|\tilde{P}|u_q(n)\rangle^2 \;=\; \sum_{k,l,m,j=0}^{L_c}\;\sum_{\sigma_k,\sigma_l,\sigma_m,\sigma_j=\pm}\langle u_p(n)|\pi_k^{\sigma_k}\tilde{P}\pi_l^{\sigma_l}|u_q(n)\rangle\,\langle u_q(n)|\pi_m^{\sigma_m}\tilde{P}\pi_j^{\sigma_j}|u_p(n)\rangle\;.$$

For the same reason as above, the sum can be restricted to elliptic channels $k, l, m, j > L_h$ up to errors of order $\mathcal{O}(\lambda)$. From the 16 signs, Lemma 1(iv) eliminates half, forcing $\sigma_k = -\sigma_l$ and $\sigma_m = -\sigma_j$. To each of the finite number of remaining summands an oscillatory sum argument will now be applied. Set

$$J(N) \;=\; \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\langle u_p(n)|\pi_k^{-\sigma}\tilde{P}\pi_l^{\sigma}|u_q(n)\rangle\;\langle u_q(n)|\pi_m^{-\tau}\tilde{P}\pi_j^{\tau}|u_p(n)\rangle\;.$$

Proceeding as above shows

$$J(N) \;=\; e^{\imath\sigma(\eta_k+\eta_l)}\,e^{\imath\tau(\eta_m+\eta_j)}\,J(N) \;+\; \mathcal{O}(N^{-1},\lambda)\;.$$

Again invoking the main hypothesis, $J(N) = \mathcal{O}(1)$ is therefore possible only if $\tau = -\sigma$ and $\{k, l\} = \{m, j\}$. Hence up to $\mathcal{O}(N^{-1}, \lambda)$,

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}\,\langle u_p(n)|\tilde{P}|u_q(n)\rangle^2 \\
&\quad= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k,l>L_h}^{L_c}\sum_{\sigma=\pm} \mathbf{E}\,\langle u_p(n)|\pi_k^{-\sigma}\tilde{P}\pi_l^{\sigma}|u_q(n)\rangle\;\langle u_q(n)|\pi_l^{\sigma}\tilde{P}\pi_k^{-\sigma}|u_p(n)\rangle &(28)\\
&\qquad+ \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k,l>L_h,\,k\neq l}^{L_c}\sum_{\sigma=\pm} \mathbf{E}\,\langle u_p(n)|\pi_k^{-\sigma}\tilde{P}\pi_l^{\sigma}|u_q(n)\rangle\;\langle u_q(n)|\pi_k^{\sigma}\tilde{P}\pi_l^{-\sigma}|u_p(n)\rangle\;. &(29)
\end{aligned}$$

Normalizing the projections of the frame vectors and then applying Lemma 1(iv) gives

$$(28) \;=\; \frac{1}{2L}\sum_{k,l>L_h}^{L_c} h_l^2\,h_k^2\,\left(\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{E}\;\rho_{p,k}(n)\,\rho_{q,l}(n)\right)\;.$$

The contribution (29) can only be treated similarly if $q = p$. Supposing this, the second identity in Lemma 1(iv) shows

$$(29) \;=\; \frac{1}{2L}\sum_{k,l>L_h,\,k\neq l}^{L_c} h_l^2\,h_k^2\,\left(\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{E}\;\rho_{p,k}(n)\,\rho_{p,l}(n)\right)\;.$$

The sum of the latter two contributions is given in (ii).

(iii) One now has to sum (28) and (29) over $q = 2L_h + 2, \ldots, L$, namely precisely the elliptic frame vectors. But because $k$ and $l$ only correspond to elliptic channels, the sum may be extended to $q = 1, \ldots, L$ because the weight of the hyperbolic frame vectors in the elliptic channels is of order $\mathcal{O}(\lambda)$ by Proposition 3. It now follows that the contribution of (29) vanishes. In order to show this, decompose $\pi_k^{\sigma}$ and $\pi_l^{\sigma}$ therein using (11) and note that the directions $|\mathrm{w}_k^{\sigma}\rangle$

and $|\mathrm{w}_l^\sigma\rangle$ on which they project satisfy the hypothesis of Lemma 2 due to the identities (10). Therefore

$$\sum_{q=1}^{L} \pi_k^\sigma |u_q(n)\rangle \, \langle u_q(n)| \pi_l^\sigma \;=\; 0 \;, \qquad k \neq l \;, \tag{30}$$

implying the claim. The sum of (28) over $q = 1, \ldots, L$ can easily be carried out using the identity (18):

$$\sum_{q=1}^{L} (28) \;=\; \frac{1}{2L} \sum_{l>L_h}^{L_c} h_l^2 \, \nu_l \sum_{k>L_h}^{L_c} h_k^2 \left( \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E} \; \rho_{p,k}(n) \right) \;.$$

Because $u_p$ is elliptic, the sum may carry over $k = 0, \ldots, L_c$ because the error is $\mathcal{O}(\lambda^2)$. □

## 4.4 Sum of Lyapunov exponents near band center

Let us again suppose in this section that there are only elliptic channels. Then it follows from (24) and Lemma 3 that

$$\begin{aligned}
\sum_{l=1}^{L} \gamma_l \;&=\; \frac{\lambda^2}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{l=1}^{L} \mathbf{E} \left( \langle u_l(n)| \, |P|^2 |u_l(n)\rangle - \frac{1}{2} \langle u_l(n)| \tilde{P} \Pi_L(n) \tilde{P} |u_l(n)\rangle \right) + \mathcal{O}(\lambda^3) \\
&=\; \lim_{N\to\infty} \frac{\lambda^2}{8} \, h_{\mathrm{av}}^2 \sum_{l=1}^{L} \sum_{k=0}^{L_c} h_k^2 \, \langle \rho_{l,k} \rangle_N \;+\; \mathcal{O}(\lambda^3)
\end{aligned}$$

Using (18), one therefore gets:

**Theorem 2** *Suppose that the Main hypothesis holds and all channels are elliptic. Then*

$$\sum_{l=1}^{L} \gamma_l \;=\; \frac{L\,\lambda^2}{8} \, \left(h_{\mathrm{av}}^2\right)^2 + \mathcal{O}(\lambda^3) \,. \tag{31}$$

Via the Thouless formula [CS], this theorem also allows to deduce a $\mathcal{O}(\lambda^2)$ correction to the density of states. However, the term linear in $\lambda$ has to be calculated separately. Theorem 2 allows to deduce an upper bound on $\gamma_L$, however, not a very tight one as we shall argue in Section 5.

**Corollary 1** *Suppose that the Main hypothesis holds and all channels are elliptic. Then there exists a constant $c$ such that*

$$\gamma_L \;\leq\; c\,\lambda^2 \, \log^2(L) \;+\; \mathcal{O}(\lambda^3) \,.$$

**Proof**. Because of the ordering of the Lyapunov exponents, it follows from Theorem 2 that

$$\gamma_L \;\leq\; \frac{\lambda^2}{8}\;\left(h^2_{\mathrm{av}}\right)^2 \;.$$

But using $\sin(\eta) \geq \frac{\eta}{\pi}$, one finds $h^2_{\mathrm{av}} \leq c_E\,\log(L)$ for some energy dependent constant $c_E > 1$. □

Finally, let us remark that it is also straight-forward to write out a perturbative formula for the top Lyapunov exponent:

$$\gamma_1 \;=\; \lim_{N\to\infty}\;\frac{\lambda^2}{4}\left[\sum_{j=0}^{L_c} h^2_{\mathrm{av}}\,h^2_j\,\langle\rho_{1,j}\rangle_N - \frac{1}{2L}\sum_{j,k=0}^{L_c} h^2_j h^2_k(2-\delta_{j,k})\,\langle\rho_{1,j}\rho_{1,k}\rangle_N\right] \;+\; \mathcal{O}(\lambda^3)\,. \tag{32}$$

This will be further analyzed in Section 5.

### 4.5 Weight of elliptic frame vectors in hyperbolic channels

It follows from Proposition 3 and the arguments in its proof that the weight of a elliptic frame vector in the hyperbolic channels is of order $\mathcal{O}(\lambda^2)$, that is for $N$ large enough

$$\langle\rho_{p,l}\rangle_N \;=\; \mathcal{O}(\lambda^2)\;, \qquad p = 2L_h+2,\ldots,L\;, \quad l = 0,\ldots,L_h\;.$$

Actually, more detailed information about the leading order term as well as the redistribution of this weight on the contracting and expanding basis vectors will be needed below. As it turns out, the elliptic frame vector is randomly kicked into the hyperbolic channels and immediately forced back out; therefore it spends (to leading order) an equal amount of time in the expanding and contracting hyperbolic directions.

**Proposition 4** *For $p = 2L_h+2,\ldots,L$ and $l = 0,\ldots,L_h$, one has*

$$\langle\rho^+_{p,l}\rangle_N \;=\; \langle\rho^-_{p,l}\rangle_N \;+\; \mathcal{O}(N^{-1},\lambda^3)\;.$$

**Proof.** Let us first calculate $\langle \mathrm{w}^\sigma_l|u_p(n+1)\rangle$ in terms of $\langle \mathrm{w}^\sigma_l|u_p(n)\rangle$ by using (15). In order to shorten the appearing expressions, let us drop the argument $n$ in $u_p(n)$. The denominator in (15) can be read off

$$\|\Lambda^p R(1-\lambda P)\,u_1\wedge\ldots\wedge u_p\|^2 \;=\; \det\left(\langle R(1-\lambda P)u_k|R(1-\lambda P)u_m\rangle_{1\leq k,m\leq p}\right) \;=\; \prod_{k=0}^{L_h} e^{2\eta_k} + \mathcal{O}(\lambda)\;.$$

Therefore $\langle \mathrm{w}^\sigma_l|u_p(n+1)\rangle$ is equal to

$$\left(\prod_{k=0}^{L_h} e^{-2\eta_k}\right)\;\langle\Lambda^{p-1}R(1-\lambda P)\,u_1\wedge\ldots\wedge u_{p-1}\wedge \mathrm{w}^\sigma_l|\Lambda^p R(1-\lambda P)\,u_1\wedge\ldots\wedge u_p\rangle\,(1+\mathcal{O}(\lambda)).$$

The appearing scalar product in $\Lambda^p\mathbb{R}^{2L}$ is given by

$$\det \begin{pmatrix} \langle R(1-\lambda P)u_k|R(1-\lambda P)u_m\rangle_{1\leq k,m\leq p-1} & \langle R(1-\lambda P)u_k|R(1-\lambda P)u_p\rangle_{1\leq k\leq p-1} \\ \langle \mathrm{w}_l^\sigma|R(1-\lambda P)u_m\rangle_{1\leq m\leq p-1} & \langle \mathrm{w}_l^\sigma|R(1-\lambda P)u_p\rangle \end{pmatrix} .$$

All the off-diagonal matrix elements are $\mathcal{O}(\lambda)$ due to the results of Section 3.5, except when $\sigma = +$. In the latter case, the entries of the lower left corner are $\mathcal{O}(1)$ for $m = 2l, 2l+1$ (again by Proposition 3). Hence the contributions to the determinant up to order $\mathcal{O}(\lambda)$ are given by the product of the diagonal elements and (in the case $\sigma = +$) by two transpositions. The diagonal elements of the upper left part are treated as above and cancel with the denominator. Therefore,

$$\begin{aligned} \langle \mathrm{w}_l^\sigma|u_p(n+1)\rangle \;&=\; e^{\sigma\eta_l}\; \langle \mathrm{w}_l^\sigma|(1-\lambda P)u_p(n)\rangle \\ &\quad - \delta_{\sigma,+}e^{-2\eta_l}\sum_{j=0,1} \langle \mathrm{w}_l^+|R(1-\lambda P)u_{2l+j}\rangle\langle R(1-\lambda P)u_{2l+j}|R(1-\lambda P)u_p\rangle \;+\; \mathcal{O}(\lambda^2)\;. \end{aligned}$$

For the case $\mathrm{w}_l^-$, one reads off

$$\langle \mathrm{w}_l^-|u_p(n+1)\rangle \;=\; e^{-\eta_l}\;\langle \mathrm{w}_l^-|(1-\lambda P)u_p(n)\rangle \;+\; \mathcal{O}(\lambda^2)\;, \tag{33}$$

while a bit of algebra invoking again Proposition 3 shows

$$\langle \mathrm{w}_l^+|u_p(n+1)\rangle \;=\; e^{-\eta_l}\;\langle \mathrm{w}_l^+|(1+\lambda P^*)u_p(n)\rangle \;+\; \mathcal{O}(\lambda^2)\;, \tag{34}$$

Now set $J^\sigma(N) = \mathbf{E}\,\frac{1}{N}\sum_{n=0}^{N-1}|\langle \mathrm{w}_l^\sigma|u_p(n)\rangle|^2$. Because $\mathbf{E}\,P = 0$, one gets by going back in history once

$$\begin{aligned} J^-(N) \;&=\; e^{-2\eta_l}\,J^-(N) + \lambda^2\,e^{-2\eta_l}\,\mathbf{E}\,\frac{1}{N}\sum_{n=0}^{N-1}|\langle \mathrm{w}_q^-|P|u_p(n)\rangle|^2 \;+\; \mathcal{O}(N^{-1},\lambda^3) \\ &=\; \lambda^2\,\frac{1}{e^{2\eta_l}-1}\,\mathbf{E}\,\frac{1}{N}\sum_{n=0}^{N-1}\langle u_p(n)|P^*|\mathrm{w}_l^-\rangle\langle \mathrm{w}_l^-|P|u_p(n)\rangle \;+\; \mathcal{O}(N^{-1},\lambda^3)\;. \end{aligned}$$

The appearing oscillatory sum can be treated by the same argument as in the proof of Lemma 3(i). This gives

$$J^-(N) \;=\; \lambda^2\,\frac{1}{4L}\,\frac{1}{e^{2\eta_l}-1}\;h_l^2\sum_{k=0}^{L_c}h_k^2\;\langle\rho_{p,k}\rangle_N \;+\; \mathcal{O}(N^{-1},\lambda^3)\;.$$

This argument can be repeated for $J^+(N)$ using (34) instead of (33). One finds $J^+(N) = J^-(N) + \mathcal{O}(N^{-1},\lambda^3)$. Now the whole argument can be repeated for $\mathrm{w}_{L-l}^\sigma$. Finally summing the contributions of $\mathrm{w}_l^\sigma$ and $\mathrm{w}_{L-l}^\sigma$ allows to conclude the proof. □

## 4.6 Perturbative formula for bottom Lyapunov exponent

The aim is now to generalize the perturbative calculation of the Lyapunov exponents given in Section 4.2 to the case where there are both elliptic and hyperbolic channels. It is convenient to introduced the scaled frame vectors:

$$\hat{u}_l(n) \;=\; e^{-\hat{\eta}_l}\, u_l(n)\;, \qquad \hat{\eta}_l \;=\; \frac{1}{2}(1-g^2_{[\frac{l}{2}]})\eta_{[\frac{l}{2}]}\;.$$

Note that, while the index on $\eta_l$ matches the channel index, the one on $\hat{\eta}_l$ matches the frame vector: for a hyperbolic frame vector, $\hat{\eta}_l$ is the expansion exponent in the direction into which $u_l$ is alined by Proposition 3, but for an elliptic frame vector $u_l$, one has $\hat{\eta}_l = 0$. Using the multilinearity of the determinant and then $\log\det_p = \mathrm{Tr}_p \log$, one finds

$$\sum_{l=1}^{p} \gamma_l - \hat{\eta}_l \;=\; \lim_{N\to\infty}\frac{1}{2N}\sum_{n=0}^{N-1} \mathbf{E}\;\mathrm{Tr}_p\left(\log\left(\langle\hat{u}_l(n)|\,|R(1-\lambda P)|^2\,|\hat{u}_k(n)\rangle_{1\le l,k\le p}\right)\right)\;.$$

The matrix elements of the leading order $|R(1-\lambda P)|^2 = R^*R + \mathcal{O}(\lambda)$ now give a unit matrix $\mathbf{1}_p$. In fact, using the orthonormality property of the frame vectors and inserting (7), one finds

$$\langle\hat{u}_l(n)|R^*R|\hat{u}_k(n)\rangle - \delta_{l,k} \;=\; \sum_{m=0}^{L_c}\sum_{\sigma=\pm}\left(e^{-\hat{\eta}_l-\hat{\eta}_k}\,e^{\sigma\eta_m(1-g_m^2)}-1\right)\,\langle u_l(n)|\pi_m^\sigma|u_k(n)\rangle \;=\; \mathcal{O}(\lambda)\;.$$

Moreover, if $l = k$ this expression is $\mathcal{O}(\lambda^2)$. Indeed, for a hyperbolic frame vector $u_l$, the summand $m = l$ has a vanishing prefactor and all the others are $\mathcal{O}(\lambda^2)$ by Proposition 3, while for elliptic $u_l$, all $m$ corresponding to elliptic channels have vanishing prefactors and all the remaining hyperbolic $m$ are $\mathcal{O}(\lambda^2)$ by Proposition 3 (actually, they were even calculated in Proposition 4). Now around $\mathbf{1}_p$ the logarithm can be expanded. In the expansion, the terms linear in $P$ can be discarded because $\mathbf{E}\,P = 0$. Hence $\sum_{l=1}^{p}\gamma_l - \hat{\eta}_l$ is up to $\mathcal{O}(\lambda^3)$ equal to

$$\lim_{N\to\infty}\frac{1}{2N}\sum_{n=0}^{N-1}\mathbf{E}\left[\sum_{l=1}^{p}(\langle\hat{u}_l(n)|R^*R|\hat{u}_l(n)\rangle-1)\;-\;\frac{1}{2}\sum_{l,k=1,\,l\neq k}^{p}\langle\hat{u}_l(n)|R^*R|\hat{u}_k(n)\rangle^2\right.$$
$$\left.+\,\lambda^2\sum_{l=1}^{p}\langle\hat{u}_l(n)|\,|RP|^2\,|\hat{u}_l(n)\rangle\;-\;\frac{\lambda^2}{2}\sum_{l,k=1}^{p}\langle\hat{u}_l(n)|(R^*RP+P^*R^*R)|\hat{u}_k(n)\rangle^2\right]\;.$$

When there are no hyperbolic channels, $R$ is orthogonal and the formula reduces to (25). The bottom exponent can now be obtained by substraction. Let us suppose that $\hat{\eta}_L = 0$ which means that $E$ is in the spectrum of $H_L(0)$:

$$
\begin{aligned}
\gamma_L \;=\; \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}\Bigg[ & \frac{1}{2}\; (\langle u_L(n)|R^*R|u_L(n)\rangle - 1) && (35)\\
& - \frac{1}{2} \sum_{l=1}^{L-1} \langle u_L(n)|R^*R|\hat{u}_l(n)\rangle^2 && (36)\\
& + \frac{\lambda^2}{2}\, \langle u_L(n)|\,|RP|^2\,|u_L(n)\rangle && (37)\\
& + \frac{\lambda^2}{4}\, \langle u_L(n)|(R^*RP + P^*R^*R)|u_L(n)\rangle^2 && (38)\\
& - \frac{\lambda^2}{2} \sum_{l=1}^{L} \langle u_L(n)|(R^*RP + P^*R^*R)|\hat{u}_l(n)\rangle^2 \Bigg] \;+\; \mathcal{O}(\lambda^3)\,. && (39)
\end{aligned}
$$

Now the terms (35) to (39), each by definition containing the average $\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E}$ but not the limit $N \to \infty$, will be treated separately. Inserting (7) and using the normalization property (18),

$$
(35) \;=\; \frac{1}{2} \sum_{l=0}^{L_c} \left(e^{(1-g_l^2)\eta_l} - 1\right)\, \langle \rho^+_{L,l}\rangle_N + \left(e^{-(1-g_l^2)\eta_l} - 1\right)\, \langle \rho^-_{L,l}\rangle_N \;=\; \sum_{l=0}^{L_h} (\cosh(2\eta_l) - 1)\, \langle \rho^+_{L,l}\rangle_N \;,
$$

the second step because the appearing averaged weights are equal by Proposition 4. Next, using orthogonality of $u_l$ and $u_L$,

$$
(36) \;=\; -\frac{1}{2N} \sum_{n=0}^{N-1} \sum_{l=1}^{L-1} \mathbf{E} \left( \sum_{k=0}^{L_c} \sum_{\sigma=\pm} \left(e^{\sigma(1-g_k^2)\eta_k} - 1\right)\, \langle u_L(n)|\pi^\sigma_k|\hat{u}_l(n)\rangle \right)^2 \;.
$$

Again, the sum over $k$ is actually restricted to the hyperbolic channels. But for hyperbolic a channel $k$, one has $\langle u_L(n)|\pi^\sigma_k|\hat{u}_l(n)\rangle = \mathcal{O}(\lambda^2)$ unless $\sigma = +$ and $l = 2k, 2k+1$ by Proposition 3. Hence

$$
(36) \;=\; -\frac{1}{2} \sum_{k=0}^{L_h} \left(e^{2\eta_k} - 1\right)^2 e^{-2\eta_k} \langle \rho^+_{L,k}\rangle_N \;+\; \mathcal{O}(\lambda^4)\;,
$$

which shows that to leading order (35) and (36) compensate. The contribution (37) was already calculated in Lemma 3(i). It will be compensated by (39) which is a bit more cumbersome to treat. Hence let us formulate it as a separate lemma.

**Lemma 4** *Let $p > 2L_h + 1$ and suppose that the Main hypothesis holds.*

$$
\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E} \sum_{l=1}^{L} \langle u_p(n)|(R^*RP + P^*R^*R)|\hat{u}_l(n)\rangle^2 \;=\; \frac{1}{2}\, h^2_{\text{av}} \sum_{l=0}^{L_c} h_l^2\, \langle \rho_{p,l}\rangle_N \;+\; \mathcal{O}(N^{-1}, \lambda)\;.
$$

**Proof.** Let us call the l.h.s. $J(N)$. Because $R^*R|u_k(n)\rangle = e^{2\hat{\eta}_k}|u_k(n)\rangle + \mathcal{O}(\lambda)$,

$$J(N) \;=\; \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E} \sum_{l=1}^{L} \langle u_p(n)|(e^{-\hat{\eta}_l}P + e^{\hat{\eta}_l}P^*)|u_l(n)\rangle \; \langle u_l(n)|(e^{\hat{\eta}_l}P + e^{-\hat{\eta}_l}P^*)|u_p(n)\rangle \;+\; \mathcal{O}(\lambda)\;.$$

Let us split $J(N) = J_h(N) + J_e(N)$ where $J_h(N)$ contains the sum over indices $l = 1, \ldots, 2L_h+1$ corresponding to hyperbolic frame vectors and $J_e(N)$ the remainder corresponding to elliptic frame vectors. In $J_h(N)$, one can replace up to $\mathcal{O}(\lambda)$

$$|u_1(n)\rangle\langle u_1(n)| \;=\; \pi_0^+\;, \qquad \sum_{j=0,1} |u_{2k+j}(n)\rangle\langle u_{2k+j}(n)| \;=\; \pi_j^+\;,$$

by Proposition 3. Hence

$$J_h(N) \;=\; \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{E} \sum_{k=0}^{L_h} \langle u_p(n)|(e^{-\eta_k}P + e^{\eta_k}P^*)\,\pi_k^+\,(e^{\eta_k}P + e^{-\eta_k}P^*)|u_p(n)\rangle \;+\; \mathcal{O}(\lambda)\;,$$

and an oscillatory sum argument implies that $J_h(N)$ is equal to

$$\sum_{m=0}^{L_c}\sum_{\sigma=\pm} \langle \rho^\sigma_{p,m}\rangle_N \sum_{k=0}^{L_h} \mathbf{E}\,\langle w^\sigma_m|(e^{-\eta_k}P + e^{\eta_k}P^*)\,\pi_k^+\,(e^{\eta_k}P + e^{-\eta_k}P^*)|w^\sigma_m\rangle \;+\; \mathcal{O}(N^{-1},\lambda)\;,$$

where $w^\sigma_m$ is some unit vector satisfying $\pi^\sigma_m w^\sigma_m = w^\sigma_m$. The expectation value of the last factor is by Lemma 1 independent of $w^\sigma_m$. Now $\langle \rho^+_{p,m}\rangle_N = \langle \rho^-_{p,m}\rangle_N + \mathcal{O}(\lambda)$ by Proposition 4 and (19). Of the four terms, the $(P, P)$ and $(P^*, P^*)$ pairs vanish after summing over $\sigma$ because of the sign in Lemma 1(ii). The remaining terms $(P, P^*)$ and $(P^*, P)$ are given by Lemma 1(iii) so that

$$J_h(N) \;=\; \frac{1}{2}\sum_{m=0}^{L_c} h_m^2\,\langle \rho^\sigma_{p,m}\rangle_N\,\frac{1}{L}\sum_{k=0}^{L_h} h_k^2\,\nu_k\,\cosh(2\eta_k) \;+\; \mathcal{O}(N^{-1},\lambda)\;.$$

Finally for the $l$ in the sum of $J_e(h)$, one has $e^{\hat{\eta}_l}P + e^{-\hat{\eta}_l}P^* = \tilde{P}$. Therefore, this is actually the term treated in Lemma 3(iii). Combining proves the lemma. $\square$

Therefore, (37) and (39) compensate to leading order just as do (35) to (36). The leading order contribution to $\gamma_L$ is thus solely given by (38). Noting that $R^*R|u_L(n)\rangle = |u_L(n)\rangle + \mathcal{O}(\lambda)$, the contribution (38) was already dealt with, in Lemma 3(ii) and we have proven:

**Theorem 3** *Suppose that $E \in \mathbb{R}$ is in the spectrum of $H_L(0)$ and satisfies the Main hypothesis. Then*

$$\gamma_L \;=\; \lim_{N\to\infty}\,\frac{\lambda^2}{8L}\sum_{j,k=0}^{L_c} h_j^2 h_k^2 (2-\delta_{j,k})\;\langle \rho_{L,j}\rho_{L,k}\rangle_N \;+\; \mathcal{O}(\lambda^3)\,. \tag{40}$$

The presented techniques also allow to write out formulas for the top Lyapunov exponent and the sum of the positive Lyapunov exponents.

**Proof** of Theorem 1. (i) follows immediately from $h_k^2 \geq 1$ and the fact that $\langle \rho_{L,j}\rho_{L,k}\rangle_N$ is a probability distribution. (ii) For hyperbolic channels $j,k = 0,\dots L_h$, the weights in (40) are $\mathcal{O}(\lambda)$ by Proposition 3 so that they can be neglected. For the remaining elliptic channels $j$, one has $h_j^2 \geq h_{L_c}^2$. But $h_{L_c}^2 = 1/\sin(\eta_{L_c}) = (1-\frac{\mu_{L_c}^2}{4})^{-1/2} = (\epsilon - \frac{\epsilon^2}{4})^{-1/2} \geq \epsilon^{-1/2}$. Replacing this concludes the proof. $\square$

# 5 More insights on the channel weights

This short section does not contain rigorous results. The aim is to get a better understanding of the averaged channel weights entering in the perturbative formulas above. Again, for simplicity, let us restrict ourselves to the situation where there are only elliptic channels. We first focus on the weights of the first frame vector $u_1$. It follows from (14) that, up to $\mathcal{O}(\lambda^3)$,

$$\begin{aligned}
\mathbf{E}\,\rho_{1,k}(n+1) - \rho_{1,k}(n) \;&=\; \lambda^2\,\mathbf{E}\Big[\langle u_1|P^*\pi_k P|u_1\rangle - \langle u_1|\pi_k|u_1\rangle\langle u_1|\,|P|^2\,|u_1\rangle \\
&\qquad + \langle u_1|\pi_k|u_1\rangle\langle u_1|\tilde{P}|u_1\rangle^2 - \langle u_1|(P^*\pi_k+\pi_k P)|u_1\rangle\langle u_1|\tilde{P}|u_1\rangle\Big]\ ,
\end{aligned}$$

where the index $n$ is left out on the r.h.s. and the expectation values is over $P(n+1)$ only. Now let us average $\mathbf{E}\,\frac{1}{N}\sum_{n=0}^{N-1}$ and suppose that the limit exists. Then the l.h.s. vanishes. Hence the coefficient of $\lambda^2$ on the r.h.s. has to vanish as well, up to $\mathcal{O}(\lambda)$. Calculating it with an oscillatory sum argument as in Section 4.3 shows that for all $k=0,\dots,L_c$:

$$\begin{aligned}
0 \;&=\; \frac{1}{2L}\sum_{l=0}^{L_c}\nu_k h_k^2 h_l^2\,\langle\rho_{1,l}\rangle - \frac{1}{2L}\sum_{l,m=0}^{L_c}\nu_m h_m^2 h_l^2\langle\rho_{1,l}\rho_{1,k}\rangle \\
&\quad + \frac{1}{2L}\sum_{l,m=0}^{L_c} h_l^2 h_m^2(2-\delta_{l,m})\,\langle\rho_{1,l}\rho_{1,m}\rho_{1,k}\rangle \;-\; \frac{1}{2L}\sum_{l=0}^{L_c} h_l^2 h_k^2(2-\delta_{k,l})\,\langle\rho_{1,l}\rho_{1,k}\rangle\,.
\end{aligned}$$

These equations give relations between the averaged first, second and third moments of the weights $\rho_{1,l}$. Analogously, one can write out equations for $\langle\rho_{1,l}\rho_{1,k}\rangle$ which then invoke up to the averaged sixth moments of the channel weights, and so on. This gives a hierarchy of equations for the channel weights. It results that the weights are independent of $\lambda$ and only depend on energy $E$ (through the the frequencies $\eta$).

It order to analyse these equations, let us close them already at first order by assuming factorization $\langle\rho_{1,l}\rho_{1,k}\rangle = \langle\rho_{1,l}\rangle\,\langle\rho_{1,k}\rangle$ and $\langle\rho_{1,l}\rho_{1,m}\rho_{1,k}\rangle = \langle\rho_{1,l}\rangle\,\langle\rho_{1,m}\rangle\,\langle\rho_{1,k}\rangle$. Furthermore we neglect the $\delta_{m,l}$ and suppose $\nu_k = 2$, both approximations which are $\mathcal{O}(1/L)$ w.r.t. the other terms. Now the sum over $l$ factors and one obtains:

$$0 \;=\; h_k^2 \;-\; \langle\rho_{1,k}\rangle \sum_{m=0}^{L_c} h_m^2 \;+\; \langle\rho_{1,k}\rangle \sum_{m=0}^{L_c} h_m^2 \,\langle\rho_{1,m}\rangle \;-\; h_k^2\,\langle\rho_{1,k}\rangle\;.$$

These equations have the unique solution (recall $h_k^2 = 1/\sin(\eta_k)$)

$$\langle\rho_{1,k}\rangle \;=\; \frac{1}{1+Z\,\sin(\eta_k)}\;,$$

where $Z \geq 0$ is such that normalization $\sum_{k=0}^{L_c}\langle\rho_{1,k}\rangle = 1$ is assured. One easily verifies that $Z \sim L$. For small $k$ (and large ones $L-k$ as well) one has $\sin(\eta_k) \sim k/L$. Hence the weight on these channels is of order of unity, while it is of order $1/L$ on the others. But the channels with small and large $k$ are precisely those near the band edges in Fig. 1(ii) where the rotation frequency is small. Hence the weight of $u_1$ is concentrated on the slowly rotating channels for which $h_k^2 = \mathcal{O}(L)$ (they can be considered to be most similar to hyperbolic channels). Hence it is expected from (32) that $\gamma_1 = c\,\lambda^2 + \mathcal{O}(\lambda^2)$ where $c = \mathcal{O}(1)$ as $L \to \infty$. Presumably, only the first few exponents are considerably larger than $\gamma_L$.

Due to symplectic blocking, the weight of $u_2$ has to be centered on slightly faster rotating channels, eccètera. In conclusion, the weight of last frame vector $u_L$ is expected to be concentrated on the channels which rotate the fastest and hence correspond to the band center. In these channels, $h_k^2 = \mathcal{O}(1)$ unless $E$ is an internal band edge in which case $h_h^2 = \infty$. Therefore, away from these points the lower bound $h_k^2 \geq 1$ which allowed to deduce Theorem 1 from Theorem 3 is presumably not so bad because the weights of $u_L$ enter into formula (40). In the case of mixed elliptic and hyperbolic channels, we expect the above argument to hold within the elliptic part of $\mathbb{C}^{2L}$, namely for the weight vectors not alined to the hyperbolic channels by Proposition 3.

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